Fractional Quantization and Fractional Quantum Hall Effect

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Abstract

We present a fractional quantization in a two dimensional space. The angular momenta of the two dimensional electrons are quantized in fractional numbers by the boundary conditions on a multi-layered Riemann surface. Extended wave functions for the incompressible quantum fluid states are presented and the cohesive and the excitation energies are given.

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Since the experimental observation of the Fractional Quantum Hall Effect(FOHE)[1] there have been a variety of theoretical approaches as well as experimental works[2,3] to understand it. This remarkable phenomenon occurs when a two dimensional electron system is under a strong magnetic field in the quantum limit $\omega_c \tau >> 1$ where $\omega_c = \frac{eB_0}{m_e c}$ is the cyclotron frequency and τ is the electronic scattering time. Hall conductance shows plateaus at certain values of the rational filling factor $\nu = \frac{n}{m}$[2,3], where n and m are integers with m being odd, i. e. $\sigma_{xy} = \frac{\nu e^2}{h}$. The filling factors for $\nu = \frac{1}{m}$ are associated with the formation of a uniform incompressible quantum fluid state so called Laughlin liquid[4]. The quasi-particles of charge $\frac{e}{m}$ are responsible for the formation of such liquid. The hierarchy states for the general fillings are proposed with the scheme of the quasi-particles[5] and the composite fermion approach[6] long time ago. However, the ground states and the excited states for the general fillings are not clear yet. In this paper, we provide the ground state wave functions for an arbitrary filling($\nu = \frac{n}{m}$) and the excited states associated with the

fractionally charged quasi-particles $\frac{e}{m}$ on a Riemann surface .

It is clear that the statistics in two dimensional space is different from those in three dimensions, because the former space is represented by the Braid group[7] while the latter is represented by the permutation group[8]. The permutation group allows only two possible statistics bosonic and fermionic statistics. However fractional statistics[5,6,9,10] is allowed in the context of the Braid group representation. The basic generator of the Braid group is the exchange of two particles in the configuration space. Consequently, if we consider the interchange of the positions of two identical particles in two dimensions, the wave function obtains a phase factor,

$$\Psi\left(z_2, z_1\right) = e^{i\theta}\Psi\left(z_1,\ z_2\right). \tag{1}$$

Here $\frac{\theta}{\pi}$ can be any real number. (i) When $\frac{\theta}{\pi}$ is an even integer or 0, the particles are bosons. (ii) When $\frac{\theta}{\pi}$ is an odd integer, the particles are fermions. (iii) For other $\frac{\theta}{\pi}$ real numbers, the particles are called anyons[10]. When $\frac{\theta}{\pi} = \frac{m}{n}$, where m and n are arbitrary integers, the particles are quantized in fractional numbers by the following arguments. Wave functions for the fractional angular momenta are multi-valued functions of the positions in the multiply connected fundamental space except for the branch points which are located at 0 and ∞. A Riemann surface for the fractional quantization is obtained by replacing the two dimensional plane with a surface made up of n sheets $R_0, R_1, \cdots, R_n$, each cut along the positive real axis with the common origin. The lower edge of the slit in the first sheet is joined to the upper edge of the slit in the second sheet with the exchange of two branch points(0, ∞), which flips the second sheet and makes the rotation in the same direction. The lower edge of the slit in the second sheet is joined to the upper edge of the slit in the third sheet in the same manner and so on until the last sheet. By joining the upper edge of the slit in the last sheet to the lower edge of the slit in the first sheet with the exchange of two branch points, we can construct an extended Riemann surface which is closed and simply connected. The geometric device for n=3 is shown in figure 1. The wave function is a continuous single-valued function of complex variables on the extended Riemann surface

with $\theta = [0, 2n\pi]$. If we continuously interchange the positions of two particles 2n times in the same direction, which corresponds to winding n times of particle 1 around particle 2, the wave function will change by a complex phase factor $e^{\theta(2n\pi)} = e^{2m\pi} = 1$. By the boundary condition of joining the wave smoothly on itself after the interchange of the particles 2n times the wave function is quantized on the extended Riemann surface. The angular momentum is quantized in integral numbers m on the n layered Riemann surface, which is associated with the fractional angular momentum $\frac{m}{n}$ in the projected two dimensional space. In the case after particle 1 interchanges n times with particle 2 in the same sense, so that 1 ends where 2 began and vice versa, the phase change $e^{\theta(n\pi)} = e^{m\pi}$. For an odd integral m, the particles are fermions.

We consider the electrons confined to the x-y plane under a transverse magnetic field $B_0\hat{z}$. Ignoring the electron-electron interaction, two electron Hamiltonian is given by

$$H = \sum_{j=1}^{2} \frac{1}{2m_e} |\frac{\hbar}{i}\nabla_j - \frac{e}{c}\mathbf{A}_j|^2, \tag{2}$$

where $\mathbf{A} = \frac{1}{2}B_0(y\hat{x} - x\hat{y})$ is the symmetric gauge vector potential. This problem separates into the center of mass coordinates and the relative coordinates given by,

$$Z = \frac{(z_1 + z_2)}{\sqrt{2}}, z = \frac{(z_1 - z_2)}{\sqrt{2}}, \tag{3}$$

where $z_j = x_j - iy_j$ is a complex number locating the j-th electron. The center of mass motion is trivial and quantized in the ordinary Landau levels with the Hamiltonian of the form

$$H_{cm} = \frac{1}{2m_e} |\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}|^2, \tag{4}$$

where the mass is m_e at the center of mass in this transformation. The Hamiltonian for the relative motion is given as

$$H_{rel} = \frac{1}{2m_e} |\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}|^2, \tag{5}$$

in two dimensional configuration space. Using the dimensionless units [energy in units of cyclotron energy $\hbar\omega_c = \hbar\frac{eB_0}{m_e c}$, length in units of magnetic length $a_0 = \left(\frac{\hbar}{m_e\omega_c}\right)^{1/2}$], we obtain

$$\frac{H_{rel}}{\hbar\omega_c} = -\frac{1}{2}\nabla^2 + \frac{i}{2}\frac{\partial}{\partial\theta} + \frac{1}{8}\rho^2, \tag{6}$$

where $\rho = (x^2+y^2)^{\frac{1}{2}}$. For the internal motion, as discussed earlier of this paper, fractional quantization $\frac{\theta}{\pi} = \frac{m}{n}$ is allowed in a two dimensional space. The magnetic flux seen by an electron is $e\oint \mathbf{A}\cdot d\mathbf{r}$ in the two dimensional configuration space. To achieve the same magnetic flux on an n layered Riemann surface the electron separates into n quasi-particles of the reduced charge $e^* = \frac{e}{n}$ and the reduced mass $m_e^* = \frac{m_e}{n}$ in each plane. The n quasi-particles in each layer are sticked together with the common origin by the property of the Rieman surface. Therefore the effective Hamiltonian for the relative coordinate is written by

$$H_{rel}^{eff} = \frac{1}{m_e^*}|\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}|^2, \tag{7}$$

on the Riemann surface. We adopt energy and length scales in which the cyclotron energy

$$\hbar\omega_c = \hbar\frac{e^*B_0}{m_e^*c} = \hbar\frac{eB_0}{m_ec}, \tag{8}$$

and the magnetic length

$$a_n = \left(\frac{\hbar}{m_e^*\omega_c}\right)^{1/2} = \sqrt{n}a_0. \tag{9}$$

In dimensionless units [energy in units of the same cyclotron energy $\hbar\omega_c$, length in units of magnetic length a_n], the effective Hamiltonian is transformed into

$$\frac{H_{rel}^{eff}}{\hbar\omega_c} = -\frac{1}{2}\nabla^2 + \frac{i}{2}\frac{\partial}{\partial\theta} + \frac{1}{8}\rho^2, \tag{10}$$

which has exactly the same form as given in eq. (6). Here the angle θ can vary from 0 to $2n\pi$ on the Riemann surface of n sheets. The Landau level wave functions are polynomials in ρ times a Gaussian function. We are therefore led to try a power series in ρ times a Gaussian function as the solution. The requirement that the power series must terminate gives the following energy eigenvalues:

$$E_{rel} = \hbar\omega_c\left(n_{rel} + \frac{1}{2}|l_{rel}| + \frac{1}{2}l_{rel} + \frac{1}{2}\right), \tag{11}$$

where n_{rel} is positive integer and l_{rel} is the angular momentum of the quasi-particles. The angular momentum l_{rel} of a quasi-particle has a fractional value $\frac{m}{n}$ in each layer, that is, each quasi-particle is quantized in fractional numbers with $\theta = [0, 2\pi]$. The angular momentum for the n degenerate quasi-particles which are sticked together on the Riemann surface will be m in the expanded space of $\theta = [0, 2n\pi]$. The value of m is even integral for a symmetric state and odd integral for an antisymmetric state. Therefore, the eigen energy spectrum represents just the ordinary Landau levels. If we consider the lowest Landau level, the eigen state for the internal motion can be obtained as

$$\phi_{\frac{m}{n}} = \frac{1}{[2n\pi 2^{\frac{m}{n}} \Gamma(\frac{m}{n})]^{1/2}} z^{\frac{m}{n}} e^{-\frac{1}{4}|z|^2}, \tag{12}$$

where the value of n in the normalization constant is due to the expansion of the angle θ from $[0, 2\pi]$ to $[0, 2n\pi]$ on the Rieman surface. The cyclotron motion in the relative coordinates has a fractional angular momentum $\frac{m}{n}$ about the origin in two dimensional space. These states (with many different values of m and n) are constructed from the statistics of the identical particles in two space dimensions and can not be derivable from any single particle state. These many particle states can now be utilized in two dimensional systems with a proper thermodynamic limit. For the fractional quantum Hall effect at a filling less than one, the wave function for the relative motion should be antisymmetric and m becomes odd integral in this case.

The model Hamiltonian for the N electrons confined in two dimensions under a transverse magnetic field can be written

$$H = \sum_{j=1}^{N} \frac{1}{2m_e} |\frac{\hbar}{i}\nabla_j - \frac{e}{c}\mathbf{A}_j|^2 + \sum_{j<k} V_{e-e}(|\mathbf{r}_j - \mathbf{r}_k|) + \sum_j V_{b-e}(\mathbf{r}_j), \tag{13}$$

where V_{e-e} is the Coulomb repulsion between electrons and V_{b-e} is the one body background potential due to a uniform density of positive charge. In the limit of high magnetic field, the electrons are at the lowest Landau level. As a result, the first term is simply a constant. For a short range interaction (shorter than r^{-2} as $r \to \infty$), the background potential is simply a constant, except close to the edges of the sample[11]. Fractional qunatum Hall

states arise from a condensation of the two dimensional electrons into a collective state, i.e., incompressible quantum fluid state, as a result of repulsive interelectron interactions[4].

We can thus construct the extended wave functions for the incompressible quantum fluid of the form[4]

$$\Psi_\mu(z_1, \cdots, z_N) = \prod_{j>k}^{N} (z_j - z_k)^\mu exp\left(-\frac{1}{4}\sum_l |z_l|^2\right), \tag{14}$$

where $\mu = \frac{m}{n}$. The states Ψ_μ are translationally invariant and eigenstates of total angular momentum. Since the total angular momentum M is a good quantum number in the projected two dimensional configuration space, we use the conventional definition of the filling factor $\nu = N(N-1)/2M$. Therefore, the wave functions Ψ_μ represent the hierarchy states for the general filling factor $\nu = \frac{n}{m}$ in FQHE. The electronic charges separate into the n degenerate layers of the Rieman surface. In each layer the quasi-particles of fundamental charge $\frac{e}{m}$ condensate into the Laughlin states. If n such degenrate layers are projected to the two dimensional space, the composite particles of charge $\frac{n}{m}e$ condensate into the incompressible quantum fluid states with a filling factor $\frac{n}{m}$. This description for the composite particle is the same as the degenerate Landau levels for the quasi-particles in the composite fermion theory propoesd by Jain[6]. For n = 1, we can recover Laughlin's results of filling factor $\frac{1}{m}$.

The ground state energy of Ψ_μ can be obtained by the ordinary hypernetted chain approximation[3,4]

$$U_{total}\left(\frac{m}{n}\right) = \frac{0.814}{\sqrt{m}}\left[\frac{0.230}{(m/n)^{0.64}} - 1\right]\frac{e^2}{a_0}. \tag{15}$$

The cohesive energy of the incompressible quantum fluid decreases as m increases. For the given m, the cohesive energy decreases as n increases: the factor $n^{0.64}$ in the first term of (15) reflects the larger magnetic length for the quasi-particles in n-layered structure.

We define the elementary excitations of Ψ_μ as

$$\Psi_\mu^{(z_0)} = \prod_{i}^{N} (z_i - z_0)^{\frac{1}{n}} \Psi_\mu. \tag{16}$$

Writing $|\Psi_\mu^{(z_0)}|^2 = e^{-H_\mu(Z_0)}$,

$$H_\mu(z_0) = -2\mu \sum_{j<k}^{N} ln|z_j - z_k| + \frac{1}{2}\sum_l |z_l|^2 + \frac{2}{n}\sum_i ln|z_i - z_0|. \tag{17}$$

The elementary excitations of Ψ_μ are particles of charge $\frac{1}{m}$ by the Berry phase calculation[12]. We can calculate the excitation energy to make the quasihole using the two-component hypernetted chain approximation[3,4]. The results for $\left(\frac{n}{m}\right) = \left(\frac{1}{3}, \frac{2}{3}\right)$; $(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$; and $(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7})$ are $(0.0095, 0.0079)\frac{e^2}{a_0}$, $(0.0018, 0.0015, 0.0013, 0.0012)\frac{e^2}{a_0}$ and $(0.00059, 0.00050, 0.00045, 0.00041, 0.00038, 0.00037)\frac{e^2}{a_0}$ respectively[13]. These results are close to the experimental results observed[14].

In conclusion, the angular momenta of the two dimensional electrons are quantized in fractional numbers on a multi-layered Riemann surface. The incompressible quantum fluid of the filling factor $\nu = \frac{n}{m}$ is related with these fractional quantum numbers. The cohesive energy of the two dimensional electron system is reduced by the expansion of the inter-electronic distance on a multi-layered Riemann surface. The effect of disorder[15] in the system is also reduced on the multi-layered Riemann surface. As a result, Hall plateau is observed for bigger values of n as m becomes large in the experiment. We believe the Hall plateau can be observed for smaller values of n with the same m observed in the experiment if the sample is cleaned to a disorder free system.

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Figure Caption

Fig. 1. Fig. (1). Geometry of three layered Riemann surface.